QUADRATIC FACTORS OF POLYNOMIALS.

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All methods of finding quadratic factors of $f(x) = \sum_0^n (a_i x^i)$ will be empirical to a certain extent; if $x^2 + mx + n$ be the factor sought, any one of the divisors of a_0 is taken as n, and its associated number m remains to be found. The older method[1] (quoted, for example, in Wentworth, McLellan and Glashan's "Algebraic Analysis") consisted in arranging in a vertical column the expressions

$$\cdots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \cdots$$

and placing after each of these numbers all its factors written with both plus and minus signs. If $r_i^{(s)}$ be one of these factors, lying in the horizontal row of $f(s)$, add s^2 to it. Then the numbers $r_i^{(s)} + s^2$ thus formed are kept on the line of $f(s)$, and constitute our actual working-table. If now an arithmetic sequence can be discovered running *vertically* through this table, the *difference* of this sequence is a provisional value of m, to be tried out by synthetic division. It is most laborious and unsatisfactory to attempt this scheme.

The method proposed by Professor Glenn, in this MONTHLY for October, 1916, locates m at the outset among the factors of a certain symmetric function of the coefficients, namely

$$P(x_i + x_j), \quad (i \neq j;\ i, j = 1, 2, 3, \cdots n);$$

the proper association of each m with its own n being then determined experimentally by synthetic division. He uses the upper limit L of the roots to cut down the number of the possible combinations, by excluding values of (m, n) that exceed $(2L, L^2)$. This is a great improvement; but there is still too much room for experiment, and the real problem of getting m as a $f(n)$ is not met. The ideal solution would be to assign a process or an equation which would directly associate with each assumed n its own m, or, at least, a minimum number of m's from which to choose. The methods presented herewith for that purpose seem to be new, and may be of interest on account of their rapid and easy application.

The Quartic. Let

$$f(x) \equiv x^4 + ax^3 + bx^2 + cx + d = (x^2 + mx + n)(x^2 + px + q).$$

Then (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq$, (4) $d = nq$. From (1) and (3), we get $m = (c - an)/(q - n)$, which from the nature of the problem must be integral. We now try every $n < L^2$, and check the m's found in (2), which, combining with (1), becomes

$$m^2 - am - (n + q - b) = 0.$$

Thus, for $\left\{\begin{matrix} & a & b & c & d \\ 1 & -6 & +3 & +22 & -6 \end{matrix}\right\}$, we find $(m, n) = (*, -1), (-4, 1),$

[1] See also Kronecker, Runge and Mandl in Crelle's *Journal*, Vols. 92, 99 and 113 respectively.

$\cdots$ and this last satisfies (2). Hence $x^2 - 4x + 1$ is a factor. Again, using $\left\{\begin{matrix} & a & & b & & c & & d \\ 1 & + & 4 & - & 4 & - & 17 & + & 10\end{matrix}\right\}$ (see Glenn, loc. cit.), we have $(m, n) = (*, 1)$, $(*, -1)$, $(*, 2)$, $(3, -2)$. This last pair satisfies (2), hence

$$f(x) = (x^2 + 3x - 2)(x^2 + x - 5).$$

For brevity and directness, this method leaves little to be desired.

The Quintic. Let

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = (x^2 + mx + n)(x^3 + px^2 + qx + r).$$

Then (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq + r$, (4) $d = nq + mr$, (5) $e = nr$. If we find q from (4) and (5), and also from (1) and (2), and compare these values, we get

$$m^2 + m\left(\frac{e}{n^2} - a\right) + \left(b - n - \frac{d}{n}\right) = 0,$$

where m is an integer. For each m found, compute p and q, and check in (3).

Using $\left\{\begin{matrix} & a & b & c & d & e \\ 1 & -1 & 6 & 9 & 1 & -4\end{matrix}\right\}$, we find for $n = \pm 1, \pm 2$, m turns out fractional or imaginary; hence there is no quadratic factor.

Again, using $\left\{\begin{matrix} & a & b & c & d & e \\ 1 & -1 & -6 & 9 & 1 & -4\end{matrix}\right\}$, for (m, n) we get $(*, 1)$, $(4, -1)$, $(-1, -1)$; and on computing p and q, the check is satisfied for $m = -1$, $n = -1$, $p = 0$, $q = -5$. Hence $f(x) = (x^2 - x - 1)(x^3 - 5x + 4)$.

The Sextic. The conditional equations are: (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq + r$, (4) $d = nq + mr + s$, (5) $e = nr + ms$, (6) $f = ns$. Hence

$$q = \frac{1}{n^3}\begin{vmatrix} d & m & 1 \\ e & n & m \\ f & 0 & n \end{vmatrix} = \begin{vmatrix} 1 & a - m \\ m & b - n \end{vmatrix},$$

and the m-eliminant is, therefore,

$$m^2\left(\frac{f}{n} - n^2\right) + m(an^2 - e) + (n^3 - n^2b + dn - f) = 0.$$

Using $\left\{\begin{matrix} & a & b & c & d & e & f \\ 1 & 2 & -5 & -7 & 8 & 3 & -2\end{matrix}\right\}$, with (3) as a check, for $n = -1$ we have $m = 2$, or -1; for $n = -2$, $m = 1$. These numbers give the three quadratic factors.

It will be noticed that the m-eliminant is a quadratic for the *quintic* and *sextic;* a cubic for the *septic* and *octic;* for a polynomial of degree $2k$ or $2k - 1$, it is of degree $k - 1$. The use of determinants makes it easy to write out these equations. One check-equation will always be left over, by means of which the

$(k-1)$ m-values can be tested. Having the m-equation, we therefore insert $n < L^2$, and look for integral roots by synthetic division, as far as $m < 2L$, where L is the upper (lower) limit of the roots of $f(x)$. We are then sure that the m's thus found are correctly paired with their own proper n.

The following table gives the m-eliminants for degrees 4 to 9; the law of their structure is obvious, so that the table can be extended indefinitely.

4-ic: $\frac{1}{n^2}\begin{vmatrix} c & m \\ d & n \end{vmatrix} - |a-m| = 0.$ Check in the b-equation.

5-ic: $\frac{1}{n^2}\begin{vmatrix} d & m \\ e & n \end{vmatrix} - \begin{vmatrix} 1 & a-m \\ m & b-n \end{vmatrix} = 0.$ Check in the c-equation.

6-ic: $\frac{1}{n^3}\begin{vmatrix} d & m & 1 \\ e & n & m \\ f & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & a-m \\ m & b-n \end{vmatrix} = 0.$ Check in the c-equation.

7-ic: $\frac{1}{n^3}\begin{vmatrix} e & m & 1 \\ f & n & m \\ g & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & a-m \\ m & 1 & b-n \\ n & m & c \end{vmatrix} = 0.$ Check in the d-equation.

8-ic: $\frac{1}{n^4}\begin{vmatrix} e & m & 1 & 0 \\ f & n & m & 1 \\ g & 0 & n & m \\ h & 0 & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & a-m \\ m & 1 & b-n \\ n & m & c \end{vmatrix} = 0.$ Check in the d-equation.

9-ic: $\frac{1}{n^4}\begin{vmatrix} f & m & 1 & 0 \\ g & n & m & 1 \\ h & 0 & n & m \\ i & 0 & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 & a-m \\ m & 1 & 0 & b-n \\ n & m & 1 & c \\ 0 & n & m & d \end{vmatrix} = 0.$ Check in the e-equation.

The General Polynomial. To find an explicit form of the m-equation for the general case, it will be expedient to use a more systematic notation, and to modify the preceding method somewhat, so as to avoid having two determinants in the same equation. Let

$$\begin{aligned} f(x) &\equiv x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0 \\ &= (x^2 + mx + n)(x^{k-2} + b_{k-3}x^{k-3} + \cdots + b_0). \end{aligned}$$

Performing the multiplication and equating coefficients, we get

$$\begin{aligned} a_{k-1} &= m + b_{k-3}, \\ a_{k-2} &= n + mb_{k-3} + b_{k-4}, \\ a_{k-3} &= nb_{k-3} + mb_{k-4} + b_{k-5}, \\ &\cdots\cdots\cdots\cdots \\ a_1 &= nb_1 + mb_0, \\ a_0 &= nb_0. \end{aligned}$$

This is a consistent set of k linear equations between the $(k-1)$ quantities $[1, b_{k-3}, b_{k-4}, \cdots, b_1, b_0]$. Eliminating these, we have at once as our m-equation

$$
\text{(I)} \qquad 0 = \begin{vmatrix} a_{k-1} & m & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{k-2} & n & m & 1 & 0 & \cdots & 0 & 0 \\ a_{k-3} & 0 & n & m & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & 0 & 0 & 0 & 0 & \cdots & n & m \\ a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & n \end{vmatrix}.
$$

This is of the $(k-1)$th degree in m, whereas the equations formerly obtained were only of degree $(k-3)/2$ or $(k-4)/2$, so that the number of m's belonging to a given n is doubled; but the present equation is more readily handled for the purpose we have in view, namely, to display the inner structure of the m-eliminant.

Let us use the symbols

$$
\Delta_2, \Delta_3, \cdots \text{ for } \begin{vmatrix} m & 1 \\ n & m \end{vmatrix}, \begin{vmatrix} m & 1 & 0 \\ n & m & 1 \\ 0 & n & m \end{vmatrix}, \cdots.
$$

Then, on continuously developing (I) in terms of its last row, we get

$$
\text{(II)} \qquad \begin{aligned} 0 &= a_0\Delta_{k-1} - na_1\Delta_{k-2} + \cdots + (-1)^{k-1}n^{k-1}a_{k-1}\Delta_0 \\ &= \sum_{\lambda=0}^{\lambda=k-1} [(-1)^\lambda n^\lambda a_\lambda \Delta_{k-\lambda-1}]; \quad (\Delta_1 = m;\ \Delta_0 = 1). \end{aligned}
$$

It remains to evaluate Δ_r. Let us write down a few of the first determinants of this form, to observe the law, noting that $\Delta_r = m\Delta_{r-1} - n\Delta_{r-2}$.

$$
\begin{aligned}
\Delta_1 &= m, & \Delta_5 &= m^5 - 4m^3n + 3mn^2, \\
\Delta_2 &= m^2 - 1n, & \Delta_6 &= m^6 - 5m^4n + 6m^2n^2 - 1n^3, \\
\Delta_3 &= m^3 - 2mn, & \Delta_7 &= m^7 - 6m^5n + 10m^3n^2 - 4mn^3, \\
\Delta_4 &= m^4 - 3m^2n + 1n^2, & \Delta_8 &= m^8 - 7m^6n + 15m^4n^2 - 10m^2n^3 + 1n^4.
\end{aligned}
$$

If one desires an explicit general formula for Δ_r it may be found as follows: The above coefficients seem to be arithmetic sequences of order 0, 1, 2, 3, $\cdots$ and a simple induction shows that this is really so. If we let $T_q^{(j)}$ stand for the *qth term of an arithmetic sequence of order j*, we have

$$
\begin{cases} \Delta_{2h} = m^{2h} - T_{2h-1}^{(1)}m^{2h-2}n + \cdots + (-1)^h T_1^{(h)} m^0 n^h, \\ \Delta_{2h+1} = m^{2h+1} - T_{2h}^{(1)}m^{2h-1}n + \cdots + (-1)^h T_2^{(h)} mn^h; \end{cases}
$$

or more briefly,

$$
\text{(III)} \qquad \Delta_r = \sum_{j=0}^{j=h} [(-1)^j T^{(j)}_{r-2j+1} m^{r-2j} n^j], \quad (r = 2h, \text{ or } 2h+1).
$$

The T's can be found from the following table:

0th order:	1	1	1	1	1	1	1	1	1	...
1st " :		1	2	3	4	5	6	7	8	...
2d " :			1	3	6	10	15	21	28	...
3d " :				1	4	10	20	35	56	...
4th " :					1	5	15	35	70	126 ...
5th " :						.	.	.	.	...

The formula for the qth term in the line j is[1]

$$T_q^{(j)} = d_1 + C_1^{q-1}d_2 + C_2^{q-1}d_3 + \cdots + C_j^{q-1}d_j.$$

In our case, the d's themselves are binomial coefficients of order j, so that

$$T_q^{(j)} = \sum_{i=0}^{i=j} (C_i^{q-1} C_i^j),$$

where $C_0^s = C_0^0 = 1$, and $C_k^s = 0$, $(k > s)$. Inserting this in (III), we obtain

$$\Delta_r = \sum_{j=0}^{j=h} \left((-1)^j \sum_{i=0}^{i=j} [C_i^{r-2j} C_i^j] m^{r-2j} n^j \right), \quad (r = 2h, \text{ or } 2h+1).$$

Finally we make use of the proposition that $T_q^{(j)} = H_j^q = C_j^{q+j-1}$, so that

$$C_s^{n-s} = \sum_{i=0}^{i=s} [C_i^{n-2s} C_i^s],$$

and get as our definite result

$$\text{(IV)} \qquad \Delta_r = \sum_{j=0}^{j=h} ((-1)^j C_j^{r-j} m^{r-2j} n^j).$$

Equations (II) and (IV) completely solve the problem of associating the n of a quadratic factor of $f(x)$ with its own proper m's, and display the nature of this association.

Incidentally, we get in (IV) the expansion of an interesting determinant. If we replace 1 by r, and put $\Delta_q \equiv D_q(n, m, r)$, we have

$$D_q(n, m, r) = \sum_{j=0}^{j=h} [(-1)^j C_j^{q-j} m^{q-2j} n^j r^j].$$

Among the special forms, we note $D_q(n, m, -1)$, which is a continuant, and $D_q(1, \pm 1, \pm 1)$, which gives us curious summation formulas.

[1] Fine, *College Algebra*, p. 365.